Nonstationarity

Some complications in the distributions

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Outline

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Asymptotic theorems

We will review the two important asymptotic theorems.

- Law of large numbers
- Central Limit Theorem

Law of large numbers

Theorem (Law of large numbers)

If $x_1, \ldots, x_n$ are iid random variables with finite mean μ and finite variance σ^2 and $\bar{X}_n = (1/n)\sum_{i=1}^{n} x_i$, then

$$\bar{X} \xrightarrow{p} \mu$$

Central Limit Theorem

Theorem (Central Limit Theorem)

If $x_1, \ldots, x_n$ are iid random variables with finite mean μ and finite variance σ^2 and $\bar{X}_n = (1/n)\sum_{i=1}^n x_i$, then

$$\bar{x}_n \xrightarrow{d} N\left(\mu, \frac{\sigma^2}{n}\right)$$

However this distribution is degenerate: the total mass is around μ. Usually, we rewrite:

$$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

Outline

Estimation

Consider the usual regression case:

$$Y = X\beta + \varepsilon$$

The OLS estimator is given by:

$$\hat{\beta} = (X'X)^{-1}X'Y$$

Inference

Under hypotheses:

- X and ε are independent
- $\varepsilon \sim iid(0, \sigma^2 I_n)$($\Leftrightarrow$ no heteroskedasticity or autocorrelation)
- $\text{plim} \frac{X'X}{n} = Q$

Proposition

- *It is unbiased*
- *Its variance is given by:* $\sigma_\varepsilon^2 (X'X)^{-1}$
- *It is convergent*
- *Its asymptotic distribution is normal.* $\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \sigma^2 Q^{-1})$

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Preliminary

To study the properties of the OLS estimator, we will start from:

Proposition

$$\hat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$$

Proof.

$$\begin{aligned}\hat{\beta} &= (X'X)^{-1}X'Y \\ &= (X'X)^{-1}X'(X\beta + \varepsilon) \\ &= \overbrace{(X'X)^{-1}X'X}^{I}\beta + (X'X)^{-1}X'\varepsilon \\ &= \beta + (X'X)^{-1}X'\varepsilon\end{aligned}$$

□

Unbiasedness of the OLS

Proposition

The OLS estimator is unbiased: $\mathsf{E}(\hat{\beta}) = \beta$

Proof.

$$\begin{aligned} \mathsf{E}[\hat{\beta}] &= \mathsf{E}\left[\beta + (X'X)^{-1}X'\varepsilon\right] \\ &= \beta + (X'X)^{-1}X'\,\mathsf{E}[\varepsilon] && \text{if } X \text{ and } \varepsilon \text{ independent} \\ &= \beta && \text{if } \mathsf{E}[\varepsilon] = 0 \end{aligned}$$

□

Variance of the OLS

Proposition

The variance of the OLS estimator is: $\text{Var}[\hat{\beta}] = \sigma_\varepsilon^2 (X'X)^{-1}$

Proof.

$$
\begin{aligned}
\text{Var}[\hat{\beta}] &= \text{Var}\left[\beta + (X'X)^{-1} X' \varepsilon\right] \\
&= (X'X)^{-1} X' \, \text{Var}[\varepsilon] \, X (X'X)^{-1} \\
&= (X'X)^{-1} X' \, \sigma_\varepsilon^2 I_n \, X (X'X)^{-1} \qquad \text{if } \text{Var}[\varepsilon] = \sigma^2 I_n \\
&= \sigma^2 \overbrace{(X'X)^{-1} X'X}^{I} (X'X)^{-1} \\
&= \sigma_\varepsilon^2 (X'X)^{-1}
\end{aligned}
$$

□

Convergence of the OLS

Proposition

The OLS estimator is convergent: $\hat{\beta} \xrightarrow{p} \beta$

Proof.

From:

$$\hat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$$

$\hat{\beta}$ can be rewritten as:

$$\hat{\beta} = \beta + \left(\frac{(X'X)}{T}\right)^{-1} \frac{X'\varepsilon}{T}$$

We will see that:

$$\left(\frac{(X'X)}{T}\right)^{-1} \xrightarrow{p} Q^{-1}$$

$$\frac{X'\varepsilon}{T} \xrightarrow{p} 0$$

□

First element: we make the assumption that: $\lim_{n\to\infty} \frac{X'X}{n} = Q$, hence $\left(\frac{(X'X)}{T}\right)^{-1} \Rightarrow Q^{-1}$

Second element: $(\frac{X'\varepsilon}{T})$

- $\mathrm{E}[X'\varepsilon] = 0$ under the assumptions:
  - X and ε independent
  - $\mathrm{E}[\varepsilon] = 0$
- $\mathrm{Var}[\frac{X'\varepsilon}{T}] = \frac{X}{T}' \mathrm{Var}[\varepsilon] \frac{X}{T} = \frac{\sigma^2}{T}\frac{X'X}{T} = \frac{\sigma^2}{T} Q \to 0$

We have hence: $\mathrm{plim}\, \frac{X'\varepsilon}{T} \to 0$

Finally, we see that:

Proposition

$$\mathrm{plim}\, \hat{\beta} = \beta + Q^{-1}0 = \beta \Rightarrow \hat{\beta} \xrightarrow{p} \beta$$

Distribution of the OLS

- Finite sample: if $\varepsilon_i \sim \mathcal{N}()$ the OLS is normally distributed
- Asymptotic: OLS is normally distributed by a TCL

Proposition

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{L} \mathcal{N}(0, \sigma^2 Q^{-1})$$

Distribution of the OLS: proof

$\hat{\beta} = \beta + \left(\frac{(X'X)}{n}\right)^{-1} \frac{X'\varepsilon}{n} \rightarrow \sqrt{n}(\hat{\beta} - \beta) = \left(\frac{(X'X)^{-1}}{n}\right) \sqrt{n}\frac{X'\varepsilon}{n}$

If:

- X and ε are independent
- $\mathrm{E}(X\varepsilon) = 0$

Define new variable $w = x_i \varepsilon_i$

We have $\bar{w} = \frac{X'\varepsilon}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i \varepsilon_i$

- Is iid
- Has expectation 0
- Has variance $\frac{\sigma^2}{n} Q$

Hence by a TCL (Lindberg-Feller): $\sqrt{n}(\bar{w} - \mathrm{E}(\bar{w})) \xrightarrow{L} \mathcal{N}(0, \sigma^2 Q)$

Proposition

$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{L} \mathcal{N}(Q^{-1}0, Q^{-1}\sigma^2 Q Q^{-1}) = \mathcal{N}(0, \sigma^2 Q^{-1})$

Review of the assumptions

We had tho make following assumptions:

- X and ε are independent
- $\varepsilon \sim iid(0, \sigma^2 I_n)$($\Leftrightarrow$ no heteroskedasticity or autocorrelation)
- $\lim_{n\rightarrow\infty} \frac{X'X}{n} = Q$

Do these assumptions hold for correlated data? (no more independent!)

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Do these assumptions hold for correlated data? (no more independent!)

Moment matrix

The assumption that plim $\frac{X'X}{n} = Q$ relies on a law of large numbers.

$$X'X = \sum_{t=1}^{T} x_t x_{t'} = \sum_{t=1}^{T} \begin{pmatrix} 1 \\ x_{1t} \\ x_{2t} \\ \vdots \\ x_{kt} \end{pmatrix} \begin{pmatrix} 1 & x_{1t} & x_{2t} & \dots & x_{kt} \end{pmatrix}$$

Theoretical moment matrix

$$Q \equiv \mathrm{E}[X'X] = \begin{pmatrix} 1 & \mu_1 & \mu_2 & \dots & \mu_k \\ \mu_1 & \sigma_1^2 + \mu_1^2 & \sigma_{12} + \mu_1\mu_2 & \dots & \sigma_{1k} + \mu_1\mu_k \\ \mu_2 & \sigma_{21} + \mu_2\mu_1 & \sigma_2^2 + \mu_2^2 & \dots & \sigma_{2k} + \mu_2\mu_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_2 & \sigma_{k1} + \mu_k\mu_1 & \dots & \dots & \sigma_k^2 + \mu_k^2 \end{pmatrix}$$

This matrix entails:

- First row or colums: Expectations of the variables
- In the diagonal: second moments (equal to the variance if $\mathrm{E}[x_i] = 0$)
- Elsewhere: second "cross-moments" (equal to the covariance if $\mathrm{E}[x_i] = E[x_j] = 0$)

Empirical moment matrix

$$X'X = \begin{pmatrix} T & \sum x_{1i} & \sum x_{2i} & \dots & \sum x_{ki} \\ \sum x_{2i} & \sum x_{2i}^2 & \sum x_{2i}x_{3i} & \dots & \sum x_{2i}x_{ki} \\ \sum x_{3i} & \sum x_{3i}x_{2i} & \sum x_{3i}^2 & \dots & \sum x_{3i}x_{ki} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum x_{ki} & \sum x_{ki}x_{2i} & \dots & \dots & \sum x_{ki}^2 \end{pmatrix}$$

Convergence of the empirical moment matrix

Theorem

$\frac{X'X}{T} \xrightarrow{p} Q$

This convergence is proved by the law of large numbers:

- $(1/n)\sum_{i=1}^{n} x_i \xrightarrow{p} \mathrm{E}[x] = \mu$
- $(1/n)\sum_{i=1}^{n} x_i^2 \xrightarrow{p} \mathrm{E}[x^2] = \mu^2 + \sigma^2$
- $(1/n)\sum_{i=1}^{n} x_{1i}x_{2i} \xrightarrow{p} \mathrm{E}[x_1 x_2] = \mu_1\mu_2 + \sigma_{12}$

When we use the fact that (Greene, p. 900, 5 ed):

Proposition

$(1/n)\sum_{i=1}^{n} g(x_i) \xrightarrow{p} \mathrm{E}[g(x)]$ *if a LLN hold for x*

Outline

Process with autocorrelation

Consider the usual AR(1) process:

$$Y_t = \varphi Y_{t-1} + \varepsilon_t$$

The OLS estimator is given by: $\hat{\varphi}_T = \frac{\sum_{i=2}^T Y_t Y_{t-1}}{\sum_{i=2}^T Y_{t-1}^2}$

It has proprieties:

Biased since the assumption that the regressors and the disturbances are independent is no more valid.

Consistent by a law of large numbers for correlated data

Normally distributed

Its asymptotic distribution is:

$$\sqrt{T}(\hat{\varphi} - \varphi) \xrightarrow{d} N(0, 1 - \varphi^2)$$

Extensions of law of large numbers and TCL

Proposition (Law of large numbers for correlated process)

If Y_t is a stationnary process with MA coefficients $\sum_{j=0}^{\infty} |\gamma_j| < \infty$, then $\bar{Y}_t \xrightarrow{p} \mu$

Proposition (TCL for martingale difference sequence)

$\frac{1}{\sqrt{T}} \sum_{i=1}^{T} \varepsilon_t Y_{t-k} \xrightarrow{L} \mathcal{N}(0, \sigma^2 \, \mathrm{E}(Y_t^2))$

So from: $\hat{\varphi}_T = \phi + \frac{\sum_{i=2}^{T} Y_{t-1}\varepsilon_t}{\sum_{i=2}^{T} Y_{t-1}^2}$

- $\sum_{i=2}^{T} Y_{t-1}^2 \xrightarrow{p} Q^{-1}$
- $\sum_{i=2}^{T} Y_{t-1}\varepsilon_t \xrightarrow{p} 0$
- $\sqrt{T} \sum_{i=2}^{T} Y_{t-1}\varepsilon_t \xrightarrow{L} \mathcal{N}(0, \sigma^2 Q)$

Covariance matrix estimation

$$\begin{aligned}\mathrm{Var}[\hat{\beta}] &= \mathrm{Var}\left[\beta + (X'X)^{-1}X'\varepsilon\right] \\ &= (X'X)^{-1}X'\,\mathrm{Var}[\varepsilon]\,X(X'X)^{-1} \\ &= (X'X)^{-1}X'\,\Omega_\varepsilon\,X(X'X)^{-1}\end{aligned}$$

So we wish to estimate: $X'\,\Omega_\varepsilon\,X$

- White estimation (HC): $S_0 = \frac{1}{n}\sum \hat{\varepsilon} x_i x_i'$
- Newey West (HAC): $S_0 + \frac{1}{n}\sum_l^L \sum_{l+1}^n w_l e_t e_{t-l}(x_t x_{t-l} + x_{t-l} x_t')$

Outline

Random walk

Recall the distribution of a AR(1) process:

$$\sqrt{T}(\hat{\varphi} - \varphi) \xrightarrow{d} N(0, 1 - \varphi^2)$$

What happens if $\phi = 1$? Zero variance? Degenerate distribution!
$\sqrt{T}(\hat{\phi} - 1) \xrightarrow{p} 0$

Definition (rate of convergence)

The rate of convergence of an estimator corresponds to the normalisation needed to ensure that it is non-degenerate.

Proposition

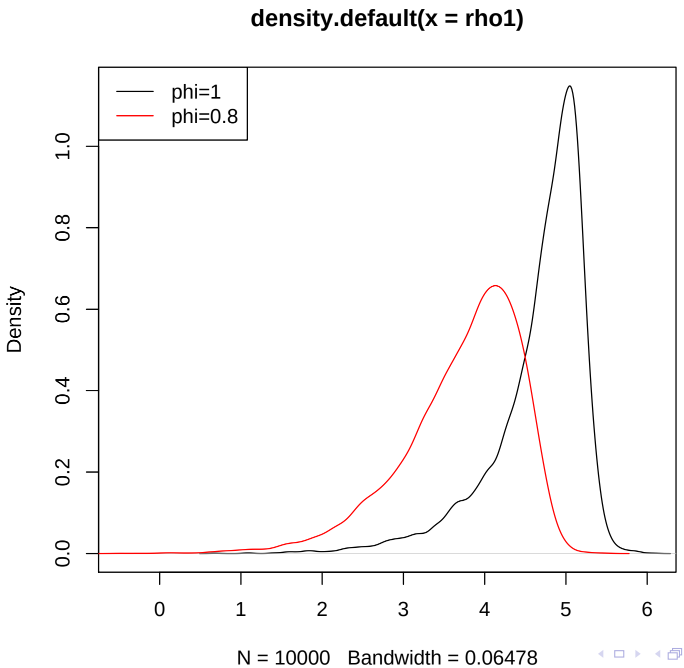
The usual rate of convergence of estimator is $\sqrt{n}$ (mean, OLS usual coefficients).

Proposition

The OLS estimator converge at rate T when $\phi = 1$. It is said super-convergent.

```
> distrho1 <- function(n) {
+     u <- rnorm(n)
+     y <- cumsum(c(0, u))
+     ylags <- embed(y, 2)
+     reg <- lm(ylags[, 1] ~ ylags[, 2] - 1)
+     sqrt(n) * (coef(reg))
+ }
> distrho <- function(n, ar) {
+     u <- rnorm(n)
+     y <- arima.sim(model = list(order = c(1, 0, 0), ar = ar),
+         n = n)
+     ylags <- embed(y, 2)
+     reg <- lm(ylags[, 1] ~ ylags[, 2] - 1)
+     sqrt(n) * (coef(reg))
+ }
> rho1 <- replicate(10000, distrho1(25))
> rho <- replicate(10000, distrho(25, 0.8))
```

```
> plot(density(rho1), xlim = range(c(rho1, rho)))
> abline(v = 10)
> lines(density(rho), col = 2)
> abline(v = 8, col = 2)
> legend("topleft", lty = 1, col = 1:2, legend = c("phi=1", "phi=0.8"))
```


density.default(x = rho1)
phi=1
phi=0.8
Density
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6
N = 10000 Bandwidth = 0.06478

Some intuition about the rate of convergence

See that each $Y_t = Y_0 + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_t \sim \mathcal{N}(0, t\sigma^2)$

The mean is:

$$\frac{1}{T}\sum_{i=1}^{T} Y_i = \overbrace{\varepsilon_1}^{Y_1} + \overbrace{\varepsilon_1 + \varepsilon_2}^{Y_2} + \ldots + \overbrace{\varepsilon_1 + \ldots + \varepsilon_T}^{Y_T}$$

$$= T\varepsilon_1 + (T-1)\varepsilon_2 + \ldots + \varepsilon_T$$

The variance of the mean is:

$$\mathrm{Var}\left(\frac{1}{T}\sum_{i=1}^{T} Y_i\right) = \frac{1}{T^2}\sum_{t=1}^{T} t^2\sigma^2 = \frac{\frac{T(T+1)(2T+1)}{6}\sigma^2}{T^2} \cong \frac{2T\sigma^2}{6}$$

Remember:

- $\sum_1^T t = \frac{T(T+1)}{2}$
- $\sum_1^T t^2 = \frac{T(T+1)(2T+1)}{6}$

So we need to normalise by $\sqrt{T}$ to obtain a stable form:

$$\frac{1}{\sqrt{T}}\bar{Y} \sim \mathcal{N}(0, \frac{1}{3}\sigma^2)$$

Distribution of $\hat{\phi}$

We need to study $T(\hat{\phi}-1)=\frac{T^{-1}\sum_{i=2}^{T}Y_tY_{t-1}}{T^{-2}\sum_{i=2}^{T}Y_{t-1}^2}$

So we find something like:

$$T(\hat{\phi}-1)\xrightarrow{L}\frac{(1/2)\{[W(1)]^2-\frac{\sigma_u^2}{\sigma^2}\}}{\int_0^1[W(r)]^2dr}$$

Definition

W(r) is a Brownian Motion. It is normally distributed, with independent variations which are also normally distributed

Differences

So when the true DGP is:

$$Y_t = Y_{t-1} + \varepsilon_t$$

And we estimate it by

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

We have the first Dickey-Fuller tests:

- $T(\hat{\phi} - 1)$
- $t_T = \frac{\hat{\phi}-1}{\hat{\sigma}_{\hat{\phi}}}$

These both tests have non-standard distributions, so critical values are needed.

Critical values finding

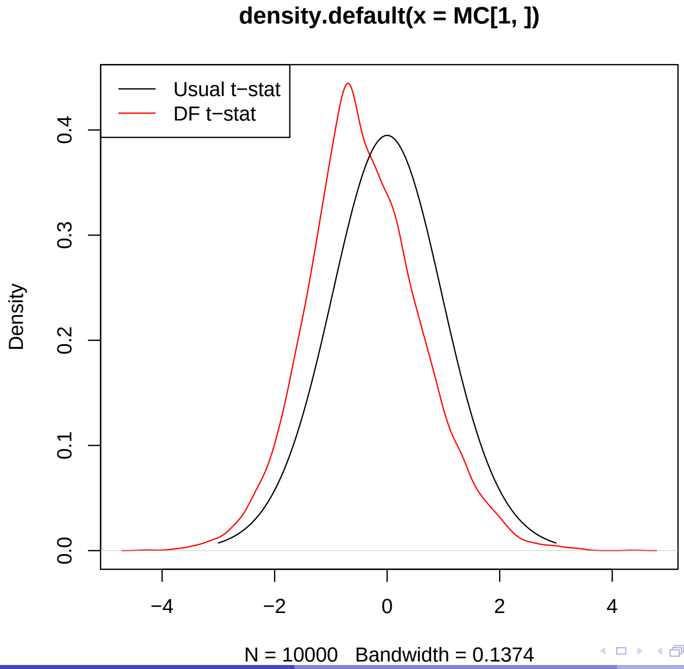
```
> tstat <- function(n) {
+     u <- rnorm(n)
+     y <- cumsum(c(0, u))
+     ylags <- embed(y, 2)
+     reg <- lm(ylags[, 1] ~ ylags[, 2] - 1)
+     tstat <- (coef(reg) - 1)/coef(summary(reg))[, "Std. Error"]
+     arstat <- n * (coef(reg) - 1)
+     return(c(tstat, arstat))
+ }
> MC <- replicate(10000, tstat(25))
```

```
> vec <- c(0.01, 0.025, 0.05, 0.1, 0.9, 0.95, 0.975, 0.99)
> round(quantile(MC[2, ], vec), 2)

    1%   2.5%     5%    10%    90%    95%  97.5%    99%
-12.14  -9.62  -7.58  -5.44   1.00   1.40   1.81   2.30

> round(quantile(MC[1, ], vec), 2)

   1%  2.5%    5%   10%   90%   95% 97.5%   99%
-2.70 -2.32 -1.98 -1.63  0.90  1.33  1.70  2.09
```

For the case $n = 25$ you find the tables:

	0.01	0.025	0.05	0.10	...	0.90	0.95	0.975	0.99
$T(\hat{\phi} - 1)$	-11.9	-9.3	-7.3	-5.3	...	1.01	1.4	1.79	2.28
$t_T = \frac{\hat{\phi}-1}{\hat{\sigma}_{\hat{\phi}}}$	-2.66	-2.26	-1.95	-1.6	...	0.92	1.33	1.7	2.16

```
> plot(density(MC[1, ]), col = 2)
> lines(curve(dt(x, df = 25), n = 100, from = -3, to = 3, add = TRUE))
> legend("topleft", lty = 1, col = c(1, 2), legend = c("Usual t-stat",
+     "DF t-stat"))
```


density.default(x = MC[1,])
Usual t−stat
DF t−stat
Density
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
N = 10000 Bandwidth = 0.1374

Generalisation to correlated errors

We saw the distribution of the $\hat{\phi}$ test to be:

$$T(\hat{\phi}-1) \xrightarrow{L} \frac{(1/2)\{[W(1)]^2-1\}}{\int_0^1 [W(r)]^2 dr}$$

But this is with iid errors, more generally it is:

$$T(\hat{\phi}-1) \xrightarrow{L} \frac{(1/2)\{[W(1)]^2-\frac{\sigma_u^2}{\sigma^2}\}}{\int_0^1 [W(r)]^2 dr}$$

Where:

- $\sigma_u^2 = \lim_{T\to\infty} T^{-1}\sum_1^T \mathrm{E}(\varepsilon_t^2)$ Variance of ε
- $\sigma^2 = \lim_{T\to\infty} T^{-1}\,\mathrm{E}(\sum_1^T \varepsilon_t)^2$

Proposition

If the erros are iid, $\sigma_u^2 = \sigma^2$

How to take into account this serial correlation?

- Obtain model with no correlation: augmented Dickey-Fuller (ADF)
- Correct the estimator to take into account the correlation: Philips

ADF test

Data is generated by an AR(p) process:

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) y_t = \varepsilon_t$$

And so we have:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t$$

It can be rewritten (Beveridge and Nelson):

$$y_t = \rho y_{t-1} + \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \ldots + \zeta_{p-1} \Delta y_{t-p-1} + \varepsilon_t$$

with $\rho = \phi_1 + \phi_2 + \ldots + \phi_p$
If there is one unit root: $\Leftrightarrow z = 1$ in $1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0$
So $\rho = 1$

ADF test

Furthermore we have the results:

Proposition

$\frac{T(\hat{\rho}-1)}{1-\hat{\zeta}_1-\hat{\zeta}_2-\ldots-\hat{\zeta}_{p-1}}$ has the same DF distribution as in the iid case..

Proposition

The t-stat has the same DF distribution as in the iid case.

Proposition

The $\hat{\zeta}_i$ have the usual Normal distribution, and hence t and F-test can be conducted in the normal way.

The PP test

Philips and Perron (1988) correct the AR(1) regression for serial correlation:

$$\hat{\phi} \text{ stat} \quad : \qquad T(\hat{\phi} - 1) - \frac{\hat{\sigma}^2 - \hat{\sigma}_u^2}{T^{-2} \sum y_{t-1}^2}$$

- $\hat{\sigma}_u^2 = T^{-1} \sum (y_t - y_{t-1})^2$
- $\hat{\sigma}^2 = T^{-1} \sum u^2 + 2T^{-1} \sum_{i=1}^{l} w_i \gamma_i$
- w_i is a weight-kernel function (Bartlett kernel as in Newey West)

Proposition

The PP correction $\hat{\phi}$ for non iid errors has the same distribution as the $\hat{\phi}$ with iid errors.

Summary

We have seen two types of tests:

- ADF: add lags in the regression (choice of p?)
- PP: correct the test for correlation (choice of kernel? of bandwidth?)

Both tests have two variants: t-test and ϕ test.

Proposition

The PP and ADF versions of the t-test and ϕ test have the same non-standard distribution, as in the iid case.

Random-walk estimated with drift

So when the true DGP is:

$$Y_t = Y_{t-1} + \varepsilon_t$$

We saw the distribution of the estimator of ϕ and of the t-test from:

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

But what if we estimate it by:

$$Y_t = \alpha + \phi Y_{t-1} + \varepsilon_t$$

Complications...

The distribution of ϕ is different, that of the t-test also, and α has non-standard distribution.

Definition (Nuisance parameter)

The α parameter is called *nuisance* parameter: its presence modifies the form of the distribution of ϕ

Case 2

We have now three hypothesis:

- $H_0 : \phi = 1$
  - DF with iid or ADF: t-test/Coefficient test
  - PP test: t-test/Coefficient test
- $H_0 : \hat{\alpha} = 0$ (not much used... PP version?)
- $H_0 : \hat{\alpha} = 0 \cap \phi = 1$

So we need four tabulated distributions:

- For t-tests
- For coefficient tests
- For t_α
- For joint hypothesis

Case 3

True DGP is:

$$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$

Estimated regression:

$$Y_t = \hat{\alpha} + \hat{\phi} Y_{t-1} + \varepsilon_t$$

But we have this time:

Proposition

$$\begin{bmatrix} T^{1/2}(\hat{\alpha} - \alpha) \\ T^{3/2}(\hat{\phi} - 1) \end{bmatrix} \xrightarrow{L} \mathcal{N}(0, \sigma^2 Q^{-1})$$

Case 3: explanation

Rewrite $Y_t = \alpha + Y_{t-1} + \varepsilon_t$:

$$y_t = y_0 + \alpha t + (u_1 + u_2 + \ldots + u_t)$$

Study the sum:

$$\sum_{i=1}^{T} y_{t-1} = \overbrace{\sum_{i=1}^{T} y_0}^{O_p(T)} + \overbrace{\sum_{i=1}^{T} \alpha(t-1)}^{O_p(T^2)} + \overbrace{\sum_{i=1}^{T} \sum_{i=1}^{T} u_i}^{O_p(T^{3/2})}$$

The regressor y_{t-1} is asymptotically dominated by the time trend $\alpha(t-1)$. In large samples, it is as if the variable y_{t-1} were replaced by the time trend $\alpha(t-1)$. (Hamilton 1994, p 497)

Case 4

True DGP is:

$$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$

Estimated regression:

$$Y_t = \hat{\alpha} + \hat{\beta} t + \hat{\phi} Y_{t-1} + \varepsilon_t$$

Complications...

The distribution of ϕ is different, that of the t-test also, α and β have non-standard distribution.

Case 4

We have many hypotheses:

- $H_0 : \hat{\phi} = 1$
  - DF with iid or ADF: t-test/Coefficient test
  - PP test: t-test/Coefficient test
- $H_0 : \hat{\alpha} = 0$ (not so used)
- $H_0 : \hat{\beta} = 0$ (not so used)
- $H_0 : \hat{\alpha} = 0 \cap \phi = 1$ (not so used)
- $H_0 : \hat{\beta} = 0 \cap \hat{\phi} = 1$ DF or ADF test

So we need for tabulated distributions:

- For t-tests (case 4)
- For coefficient tests (case 4)
- For joint hypothesis

Case 5

Case 5 is not in Hamilton 1994 (but see Pfaff 2007)
True DGP is:

$$Y_t = \alpha + \beta t + Y_{t-1} + \varepsilon_t$$

Estimated regression:

$$Y_t = \hat{\alpha} + \hat{\beta} t + \hat{\phi} Y_{t-1} + \varepsilon_t$$

Proposition

The distribution of the parameters is normal

Again, the deterministic trend dominates the stochastic one.

Interpretation of parameters

Interpretation

The interpretation/effect of the parameters is different under H_0 and H_1!

Take case 3: True DGP is:

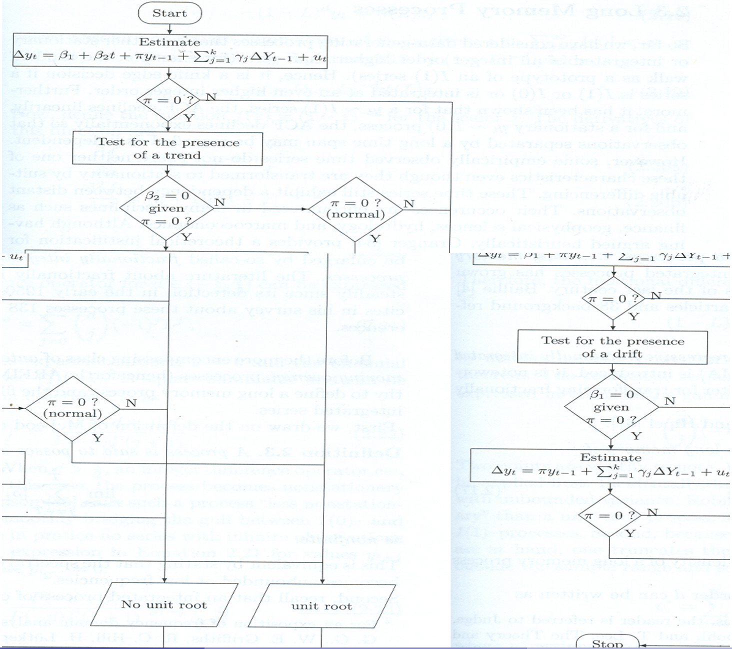
$$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$

Estimated regression:

$$Y_t = \hat{\alpha} + \hat{\phi} Y_{t-1} + \varepsilon_t$$

α is:

- Under H_0: a trend parameter ($Y_t = at + Y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i}$)
- Under H_1 a level parameter ($Y_t = \frac{a}{1-\varphi} + b \sum_{i=0}^{t-1} \varphi^i (t-i) + \sum_{i=0}^{t-1} \varphi^i \varepsilon_{t-i}$)


Start
Estimate
$\Delta y_t = \beta_1 + \beta_2 t + \pi y_{t-1} + \sum_{j=1}^{k} \gamma_j \Delta Y_{t-1} + u_t$
$\pi = 0$?
N
Y
Test for the presence of a trend
$\beta_2 = 0$ given $\pi = 0$?
N
Y
$\pi = 0$? (normal)
N
Y
$\pi = 0$?
N
Y
Test for the presence of a drift
$\beta_1 = 0$ given $\pi = 0$?
N
Y
Estimate
$\Delta y_t = \pi y_{t-1} + \sum_{j=1}^{k} \gamma_j \Delta Y_{t-1} + u_t$
$\pi = 0$?
N
Y
$\pi = 0$? (normal)
N
Y
No unit root
unit root
Stop

Size and power problem

Recall:

Definition

Size of a test The *nominal* size of a test is the theoretical probability to reject (take as false) a true event (should not).

This is the α error, fixed at $5\%, 10\%$...
However the empirical size can be higher than observed!

Size with a pure RW process

```
> library(urca)
> ur.rw <- function(n = 100) {
+     a <- cumsum(c(0, rnorm(n)))
+     ur.df(a)@teststat
+ }
> rep <- replicate(1000, ur.rw())
> mean(ifelse(rep < -1.6, 1, 0))
```

```
[1] 0.109
```

Size with a an ARIMA(0,1,1)

```
> ur.IMA <- function(n, a, test = ur.df) {
+     e <- rnorm(n)
+     pr <- (1 + a) * cumsum(e) - a * e[n]
+     test(pr)@teststat
+ }
> rep2 <- replicate(1000, ur.IMA(100, a = 0.3))
> mean(ifelse(rep2 < -1.6, 1, 0))
```

```
[1] 0.107
```

```
> rep3 <- replicate(1000, ur.IMA(100, a = -0.9))
> mean(ifelse(rep3 < -1.6, 1, 0))
```

```
[1] 0.108
```

```
> rep4 <- replicate(1000, ur.IMA(100, a = 1.2, test = ur.pp))
> mean(ifelse(rep4 < -1.6, 1, 0))
```

```
[1] 0.817
```

Power of the tests

```
> ur.ar <- function(n, ar) {
+     ar <- arima.sim(model = list(model = c(1, 0, 0), ar = ar),
+         n = n)
+     ur.df(ar)@teststat
+ }
> rep5 <- replicate(1000, ur.ar(100, 0.99))
> mean(ifelse(rep5 < -1.6, 1, 0))
```

```
[1] 0.224
```

```
> rep6 <- replicate(1000, ur.ar(100, 0.9))
> mean(ifelse(rep6 < -1.6, 1, 0))
```

```
[1] 0.923
```

Choice of the lag order

ADF test recquires choosing p.
Recall that

Proposition

The $\hat{\zeta}_i$ have the usual Normal distribution, and hence t and F-test can be conducted in the normal way.

- Sequential t-test procedure
- Information based rule: AIC, BIC
- Some rule: $k = \left[c(\frac{T}{100})^{1/d}\right]$

Observations show:

- AIC BIC choose too much

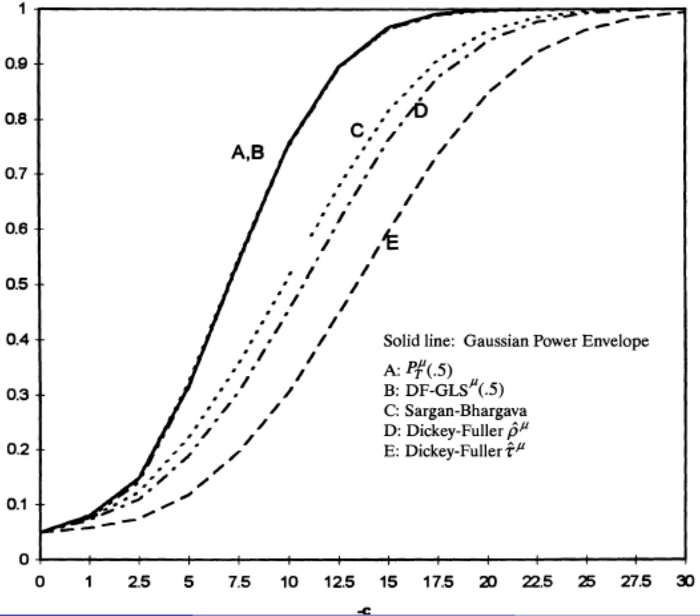
ERS test

DF-GLS test:

- Stock (1994) showed that there is no uniformly more powerful test.
- Obtain power envelope by Neyman-Pearson lemma: no test can be better, for fixed α error, than this envelope.
- See that in case without constant or trend, usual tests reach this bound
- In cases with mean and trend, tests are far below
- Conclusion: detrend the data (with GLS) and apply then ADF t-test.

P-test:
Other procedure but gives almost same results as DF-GLS.

ERS power envelope with $c = T(\phi - 1)$



Outline

KPSS test

KPSS (1992): H_0 is stationarity

- Level stationarity I(0)
- Trend stationarity not I(0) but not I(1)!

$$y_t = \alpha t + r_t + \varepsilon_t$$

Paramater constancy:

$$r_t = r_{t-1} + u_t$$

H_0 : Var$(u) = 0$ so r is a constant $\Rightarrow$ y_t is stationnary in level/trend

KPSS 2

LM test statistic:

$$\frac{\sum_{t=1}^{T}(\sum_{i=1}^{t}\varepsilon_i)^2}{\hat{\sigma}_\varepsilon^2}$$

With iid errors: Take simple estimator of the variance of ε

With non iid errors: σ_ε^2 is estimated as in PP test:

$$\tilde{\sigma}_\varepsilon^2 = T^{-1}\sum u^2 + 2T^{-1}\sum_{i=1}^{l} w_i\gamma_i$$

and the kernel/weight function is the Bartlett window: $w(l,s) = 1 - \frac{s}{l+1}$

KPSS test 3

Simulation show:

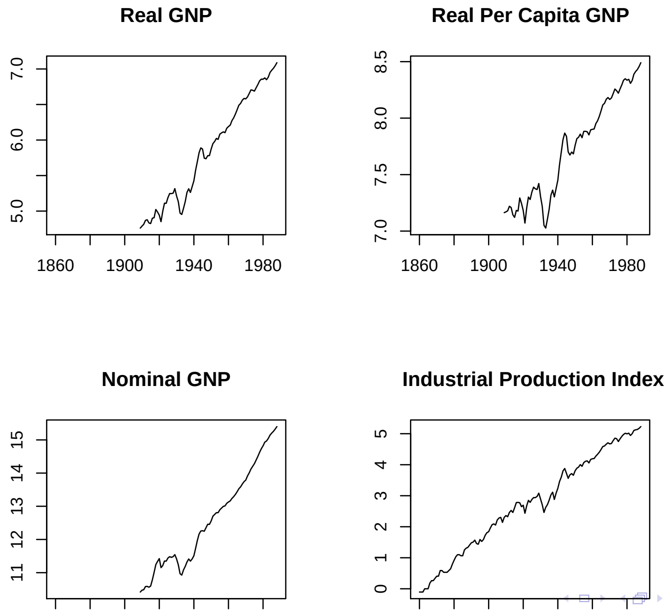
- Considerable size distortion when ther errors follow AR(1)
- Power is very low when l is big (12)
- Increasing l decreases power

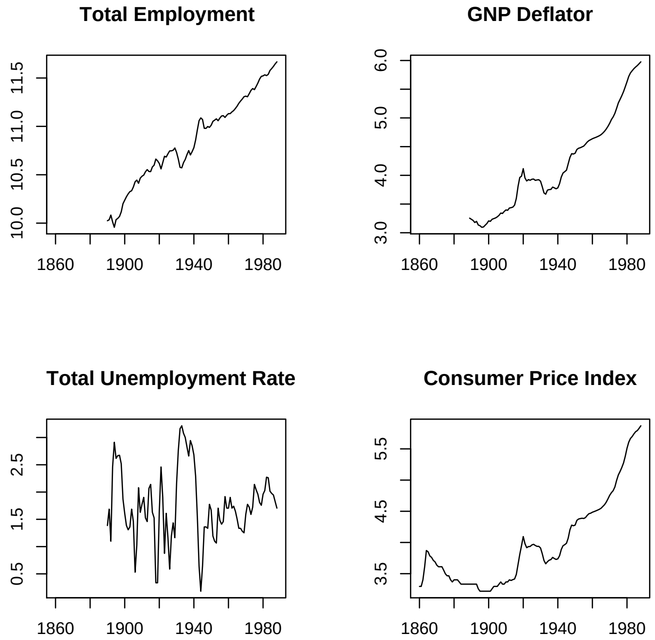
Nelson and Plosser (1982) study

Nelson and Plosser (1982) investigate 14 time series:

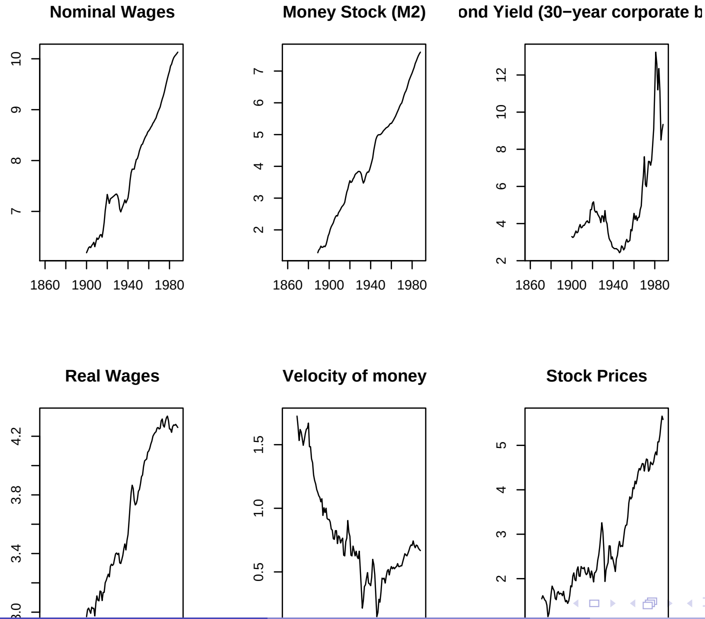
- Real GNP
- Nominal GNP
- Real Per Capita GNP
- Industrial Production Index
- Total Employment
- Total Unemployment Rate
- GNP Deflator
- Consumer Price Index
- Nominal Wages
- Real Wages
- Money Stock (M2)
- Velocity of money
- Bond Yield (30-year corporate bonds)
- Stock Prices

Nelson and Plosser (1982) study




Total Employment
GNP Deflator
Total Unemployment Rate
Consumer Price Index
1860
1900
1940
1980

Nelson and Plosser (1982) study 3



Results of the test of Trend stationary vs Difference stationary:

NP Results

13 series can be viewed as DS, one (unemployment) as TS.

KPSS results

- 1 series is level stationary
- 4 series are I(1): reject stationarity (at every $l = 1 \dots 8$ and don't reject unit root
- 3 series seem to be I(1) (result depends on l)
- 6 series: can't reject either the unit root or the trend stationnary H_0, *the conclusion is that the data are not sufficiently informative.*

Choose want you want

For 10 series, the result can be interpeted as I(1) or stationary around trend... up to you!

Outline

Packages

- Urca: ADF, PP, ERS, KPSS
- fUnitRoots: ADF with McKinnon (1996) critical values
- uroot: ADF (with AIC, BIC, t-stat procedure), seasonal unit roots: HEGY and Hansen & Canova

Missing:Ng & Perron Test, which seems to have good size and high power.

Running this sweave+beamer file

To run this Rnw file you will need:

- Package urca
- ERS.png and table.pdf in file Datasets
- lect4UnitRoot-002.eps/pdf and lect4UnitRoot-002.eps/pdf in Datasets. Those can be actually run from the code but have been saved to avoid too many computations every time.
- (Optional) File Sweave.sty which change output style: result is in blue, R commands are smaller. Also in same folder as .Rnw file.